11.8: 10, 12, 16, 24, 38

10. $f(x,y,z) = x^2y^2z^2$, $g(x,y,z) = x^2+y^2+z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $\nabla f = \lambda\nabla g$ implies (1) $\lambda = y^2z^2 = x^2z^2 = x^2y^2$ and $\lambda \neq 0$, or (2) $\lambda = 0$ and one or two (but not three) of the coordinates are 0. If (1) then $x^2 = y^2 = z^2 = \frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

12. $f(x,y,z) = x^4+y^4+z^4$, $g(x,y,z) = x^2+y^2+z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 4x^3, 4y^3, 4z^3\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$.
Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$ then $\nabla f = \lambda\nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ yielding 8 points each with an f-value of $\frac{1}{3}$.
Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.
Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1. Thus on $x^2+y^2+z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

16. $f(x,y,z) = 3x - y - 3z$, $g(x,y,z) = x + y - z = 0$, $h(x,y,z) = x^2 + 2z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 3, -1, -3\rangle$, $\lambda\nabla g = \langle \lambda, \lambda, -\lambda\rangle$, $\mu\nabla h = (2\mu x, 0, 4\mu z)$. Then $3 = \lambda + 2\mu x$, $-1 = \lambda$ and $-3 = -\lambda + 4\mu z$, so $\lambda = -1$, $\mu z = -1$, $\mu x = 2$. Thus $h(x,y,z) = 1$ implies $\dfrac{4}{\mu^2} + 2\left(\dfrac{1}{\mu^2}\right) = 1$ or $\mu = \pm\sqrt{6}$, so $z = \mp\frac{1}{\sqrt{6}}$; $x = \pm\frac{2}{\sqrt{6}}$; and $g(x,y,z) = 0$ implies $y = \mp\frac{3}{\sqrt{6}}$. Hence the maximum of f subject to the constraints is $f\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) = 2\sqrt{6}$ and the minimum is $f\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) = -2\sqrt{6}$.

24. Let $f(x,y,z) = s(s-x)(s-y)(s-z)$, $g(x,y,z) = x + y + z$. Then $\nabla f = \langle -s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y)\rangle$, $\lambda\nabla g = \langle \lambda, \lambda, \lambda\rangle$. Thus (1) $(s-y)(s-z) = (s-x)(s-z)$ and (2) $(s-x)(s-z) = (s-x)(s-y)$. (1) implies $x = y$ while (2) implies $y = z$, so $x = y = z = p/3$ and the triangle with maximum area is equilateral.

38. Let the dimensions of the box be x, y, and z, so its volume is $f(x,y,z) = xyz$, its surface area is $g(x,y,z) = xy + yz + xz = 750$ and its total edge length is $h(x,y,z) = x + y + z = 50$. Then $\nabla f = \langle yz, xz, xy\rangle = \lambda\nabla g + \mu\nabla h = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y)\rangle + \langle \mu, \mu, \mu\rangle$. So (1) $yz = \lambda(y+z) + \mu$, (2) $xz = \lambda(x+z) + \mu$, and (3) $xy = \lambda(x+y) + \mu$. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then (1) minus (2) implies $z(y-x) = \lambda(y-x)$ or $\lambda = z$, and (1) minus (3) implies $y(z-x) = \lambda(z-x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence $50 - 2\lambda = \dfrac{750 - \lambda^2}{2\lambda}$ or $3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}(50 \mp 10\sqrt{10}), \frac{1}{3}(50 \pm 5\sqrt{10}), \frac{1}{3}(50 \pm 5\sqrt{10})\right)$.

Thus the minimum of f is $f\left(\frac{1}{3}(50 - 10\sqrt{3}), \frac{1}{3}(50 + 5\sqrt{10}), \frac{1}{3}(50 + 5\sqrt{10})\right) = \frac{1}{27}(87{,}500 - 2500\sqrt{10})$, and its maximum is $f\left(\frac{1}{3}(50 + 10\sqrt{10}), \frac{1}{3}(50 - 5\sqrt{10}), \frac{1}{3}(50 - 5\sqrt{10})\right) = \frac{1}{27}(87{,}500 + 2500\sqrt{10})$.

Note: If either y or z is the distinct side, then symmetry gives the same result.